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# LINEAR MULTISTEP METHODS WITH MINIMIZED TRUNCATION ERROR FOR PERIODIC INITIAL VALUE PROBLEMS

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**Linear multistep methods with minimized truncation error for periodic initial value problems** *

by

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ABSTRACT

A common feature of most methods for numerically solving ordinary differential equations is that they consider the problem as a standard one without exploiting specific properties the solution may have.

Here we consider initial value problems the solution of which is a priori known to possess an oscillatory behaviour. The methods are of linear multistep type and special attention is paid to minimization of those terms in the local truncation error which correspond to the oscillatory solution components. Numerical results obtained by these methods are reported and compared with those obtained by the corresponding conventional linear multistep methods and by the methods developed by Gautschi.

KEY WORDS & PHRASES: *periodic initial value problems, linear multistep methods, accuracy*

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* This report will be submitted for publication elsewhere.

## 1. Introduction

We will be concerned with linear $k$-step methods

$$\rho(E)y_n - h\sigma(E)\dot{y}_n = 0 \tag{1.1}$$

for integrating the equation

$$\dot{y}(t) = f(t, y(t)) \tag{1.2}$$

in cases where the exact, local solution is known to be approximately of the form

$$y(t) \approx c_0 + \sum_{j=1}^{m} c_j e^{i\omega_j t}, \tag{1.3}$$

where the frequencies $\omega_j$ are in the interval $[\underline{\omega}, \overline{\omega}]$ with $\underline{\omega}$ and $\overline{\omega}$ given numbers. Assuming (1.3), the values of $\underline{\omega}$ and $\overline{\omega}$ can often be derived from the eigenvalue spectrum of the Jacobian matrix $\partial f / \partial y$ (see e.g. Section 4).

In the special case where the frequencies $\omega_j$ in (1.3) are such that the solution is periodic or "almost" periodic, that is $y(t) \approx y(t + 2\pi / \omega_0)$ for some a priori given frequency $\omega_0$, Gautschi [2] has developed special linear multistep methods. However, these methods are rather sensitive to a correct prediction of the frequence $\omega_0$ (cf. Sections 4.2 and 4.3) This unfavourable property of the Gautschi methods (which are of Adams type, i.e. $\rho(\zeta) = \zeta^k - \zeta^{k-1}$) motivated Neta and Ford [7] to propose methods of Nyström and Milne-Simpson type ($\rho(\zeta) = \zeta^k - \zeta^{k-2}$). These methods, however, although demonstrating a less sensitive behaviour if the value of $\omega_0$ is perturbed, are rather sensitive to non-imaginary noise (cf. Section 4.3).

In this paper we will try to construct methods which do not suffer the above mentioned disadvantages.

## 2. Minimization of the truncation error for a given interval of frequencies

Let

$$\phi(z) := \rho(e^z) - z\,\sigma(e^z), \tag{2.1}$$

then the local truncation error at $t_{n+k}$ is given by (cf. e.g. Lambert [4, p. 27])

$$\mathbf{T}_{n+k} := \phi(h\frac{d}{dt})y(t)|_{t=t_n}, \tag{2.2}$$

where $y(t)$ denotes the exact solution satisfying $y(t_n)=y_n$. We assume $\phi(0)=0$; in the case (1.3) we then have approximately

$$|\mathbf{T}_{n+k}| \leqslant \sum_{j=1}^{m} |c_j|\;|\phi(i\nu_j)|, \quad \nu_j := \omega_j h. \tag{2.3}$$

In the case where $y(t)$ is a periodic or "almost" periodic function with frequency $\omega_0$, we may replace $y(t)$ by the approximating Fourier series

$$y(t) = \sum_{l=0}^{\infty} \tilde{c}_l e^{il\omega_0 t} + \epsilon(t), \quad |\epsilon| << 1 \tag{2.4}$$

to obtain the approximate inequality

$$|\mathbf{T}_{n+k}| \leqslant \sum_{l=0}^{\infty} |\tilde{c}_l|\;|\phi(il\nu_0)|, \quad \nu_0 := \omega_0 h. \tag{2.5}$$

The inequalities (2.3) and (2.5) suggest essentially three approaches for adapting the linear multistep method to the additional information available on the exact, local solution $y(t)$. Let us start with a family of linear $k$-step methods containing $2q$ not yet specified coefficients, and let the remaining coefficients be such that $\phi(z)=O(z^r)$, $r \geqslant 1$. Then one may proceed as indicated in Table 2.1.

The *first approach* is that of Gautschi. The resulting method is said to be of *trigonometric* order $q$. Its order in the conventional sense (the *algebraic* order) is given by $p=2q$. The Gautschi method may be interpreted as a method which is *exponentially fitted* at the points $il\omega_0$ ,$l=1,...,q$ (cf. Liniger and Willoughby [5]).

**Table 2.1.** Minimization of truncation errors

| | | |
|---|---|---|
| I | $y(t)$ is periodic with frequency $\omega_0$: | Solve the system (Gautschi [2]) $\phi(il\omega_0 h)=0, \quad l=1,...,q$ |
| II | $y(t)$ has dominant solution components $exp(i\omega_j t)$ with given $\omega_j$: | Solve the system $\phi(i\omega_j h)=0, \quad j=1,...,q$ |
| III | $y(t)$ has dominant solution components $exp(i\omega t)$ with $\omega \in [\underline{\omega}, \overline{\omega}]$: | minimize the function (cf. (2.7)) $|\phi(i\omega h)|$ on $\underline{\omega} \leqslant \omega \leqslant \overline{\omega}$ |

Many anthors proposed integration methods following the *second approach.* For example, Stiefel and Bettis [9], Bettis [1] and Lyche [6] constructed schemes by which not only the harmonic oscillations

$\omega_j$ are integrated exactly but by which also products of Fourier and ordinary polynomimals are integrated without truncation error. However, to apply this approach we should start whith a linear multistep method containing sufficiently many free parameters in order to achieve that $\phi(i\omega_j h)=0$ for all $\omega_j$ occuring in (1.3), i.e. $q \geqslant m$. Another disadvantage is that a rather detailed knowledge of the dominant solution components is required. And even if this information is available, the frequencies $\omega_j$ may vary over one integration step (e.g. in non-linear problems) which will decrease the accuracy of these methods. Therefore, we are automatically led to the last approach of Table 2.1.

This *third approach* seems to be applicable to a fairly large class of problems. However, we have first to solve the minimax problem in which the $2q$ parameters in $\phi$ are to be determined in such a way that $\max|\phi(i\nu)|$ is minimal in the interval $\underline{\nu} \leqslant \nu \leqslant \overline{\nu}$ (with $\underline{\nu} := \underline{\omega} h$, $\overline{\nu} := \overline{\omega} h$). Basically, this is the problem of assigning optimal zeros to $\phi(i\nu)$ in the interval $\underline{\nu} \leqslant \nu \leqslant \overline{\nu}$. We will approximately solve this problem for small values of $\overline{\nu}-\underline{\nu}=(\overline{\omega}-\underline{\omega})h$. Let the family of linear $k$-step methods containing the $2q$ free coefficients be such that $\phi(z)=O(z^r)$, $r \geqslant 1$. Then, for sufficiently small values of $\overline{\nu}-\underline{\nu}$, we approximate $|\phi(i\nu)|$ by

$$|\phi(i\nu)| \;\approx\; \nu^r P_\mu(\nu) \approx \left(\frac{\overline{\nu}+\underline{\nu}}{2}\right)^r P_\mu(\nu), \quad \nu \in [\underline{\nu},\overline{\nu}], \tag{2.6}$$

where $P_\mu$ is a polynomial of degree $\mu$ in $\nu$ assuming nonnegative values in the interval $[\underline{\nu},\overline{\nu}]$. Evidently, any zero of $\phi(i\nu)$ corresponds to a double zero of $P_\mu(\nu)$. Since the $2q$ free coefficients allow only $q$ zeros of $\phi(i\nu)$ in the interval $[\underline{\nu},\overline{\nu}]$ we choose $\mu=2q$ and we look for a nonnegative polynomial $P_{2q}(\nu)$ with $q$ double zeros and with a minimal maximum norm in the interval $[\underline{\nu},\overline{\nu}]$. If we can find a polynomial satisfying the so-called "equal ripple property", we have found the optimal polynomial $P_{2q}(\nu)$. Now consider the shifted Chebyshev polynomial

$$\alpha\left[1+T_{2q}\left(\frac{2\nu-\overline{\nu}-\underline{\nu}}{\overline{\nu}-\underline{\nu}}\right)\right], \quad T_{2q}(x) := \cos[2q \ \arccos\ x],$$

where $\alpha$ is an appropriate constant. It is easily verified that this polynómial satisfies the equal ripple property in $[\underline{\nu},\overline{\nu}]$ so that the optimal polynomial $P_{2q}(\nu)$ is of this form. As a consequence, $P_{2q}(\nu)$ has the $q$ zeros

$$\nu^{(l)} := \frac{1}{2}(\overline{\nu}+\underline{\nu})+\frac{1}{2}(\overline{\nu}-\underline{\nu})\cos\frac{2l-1}{2q}\pi, \quad l=1,2,\ldots,q. \tag{2.7a}$$

The free coefficients in the function $\phi$ are therefore determined by the (linear) system

$$\phi(i\nu^{(l)}) \;=\; 0, \quad l=1,2,\ldots,q. \tag{2.7b}$$

The assumption that we started with a method in which $\phi(z)$ has a zero of order $r$ at $z=0$, and the condition (2.7), together imply that $\phi(z)$ has a zero of order $r+2q$ at $z=0$. Thus, the method resulting from the third approach is of (algebraic) order $p=r+2q-1$.

In this paper we concentrate on methods satisfying the minimax conditions (2.7). These methods will be called *minimax methods*. In particular we will consider the minimax methods generated by

$$\begin{aligned}
&\{k=5,\ \rho(\zeta)=\zeta^5-\zeta^4,\ r=1,\ q=3,\ \sigma(\zeta) \text{ determined by (2.7a) and (2.7b)}\},\\
&\{k=5,\ \rho(\zeta)=\zeta^5-\zeta^3,\ r=1,\ q=3,\ \sigma(\zeta) \text{ determined by (2.7a) and (2.7b)}\},\\
&\{k=6,\ 147\sigma(\zeta)=60\zeta^6,\ r=1,\ q=3,\ \rho(\zeta) \text{ determined by (2.7a) and (2.7b)}\}.
\end{aligned} \tag{2.8}$$

These methods will be denoted by $AM_6(\underline{\nu},\overline{\nu})$, $MS_6(\underline{\nu},\overline{\nu})$ and $BD_6(\underline{\nu},\overline{\nu})$, respectively. For $h \to 0$ they successively converge to the well-known 6-th order Adams-Moulton ($AM_6$), Milne-Simpson ($MS_6$) and Backward Differentiation ($BD_6$) method. These conventional methods are characterized by

$$AM_6: \rho(\zeta)=\zeta^5-\zeta^4;\quad 1440\sigma(\zeta)=475\zeta^5+1427\zeta^4-798\zeta^3+482\zeta^2-173\zeta+27,$$
$$MS_6: \rho(\zeta)=\zeta^5-\zeta^3;\quad 90\sigma(\zeta)=28\zeta^5+129\zeta^4+14\zeta^3+14\zeta^2-6\zeta+1, \tag{2.9}$$
$$BD_6: 147\rho(\zeta)=147\zeta^6-360\zeta^5+450\zeta^4-400\zeta^3+225\zeta^2-72\zeta+10;\quad 147\sigma(\zeta)=60\zeta^6,$$

respectively.

Crucial for the accuracy behaviour of the minimax methods and of the conventional methods as well, is the maximum norm of the corresponding function $\phi(z)$, where $z \in [i\underline{\nu}, i\overline{\nu}]$, i.e.

$$M(\underline{\nu},\overline{\nu};q) := \max_{\underline{\nu}\leq\nu\leq\overline{\nu}} |\phi(i\nu)|. \tag{2.10}$$

For several $(\underline{\nu},\overline{\nu})$-intervals we calculated the $M(\underline{\nu},\overline{\nu};3)$-values for the minimax methods (2.8) and compared them with $M(0,\overline{\nu};0)$, which determines the local truncation error of the corresponding conventional method. In Table 2.2 we give these $M(0,\overline{\nu};0)$-values for the methods (2.9), while Table 2.3 contains the gain factors

**Table 2.2.** $M(0,\overline{\nu};0)$-values for the conventional methods (2.9)

| $\overline{\nu}$ | $AM_6$ | $MS_6$ | $BD_6$ |
|---|---|---|---|
| .05 | $.11\ 10^{-10}$ | $.76\ 10^{-11}$ | $.46\ 10^{-10}$ |
| .10 | $.14\ 10^{-8}$ | $.98\ 10^{-9}$ | $.58\ 10^{-8}$ |
| .15 | $.24\ 10^{-7}$ | $.17\ 10^{-7}$ | $.99\ 10^{-7}$ |

**Table 2.3.** $M(0,\overline{\nu};0)\,/\,M(\underline{\nu},\overline{\nu};3)$ factor for the minimax methods (2.8)

| $\overline{\nu}$ | $\underline{\nu}=0$ | $\underline{\nu}=.05$ | $\underline{\nu}=.10$ |
|---|---|---|---|
| .05 | 10 | $\infty$ | |
| .10 | 10 | 48 | $\infty$ |
| .15 | 10 | 24 | 140 |

$M(0,\overline{\nu};0)\,/\,M(\underline{\nu},\overline{\nu};3)$ for the minimax methods (2.8). These factors turned out to be constant for the several types of methods.

## 3. Stability

Concerning the stability of the linear multistep methods we followed the usual approach as can be found in e.g. [4]. In Figure 3.1 we plotted (parts of) the stability regions of the $AM_6$, $MS_6$ and $BD_6$ method (we mention that the regions are symmetric about the real axis).

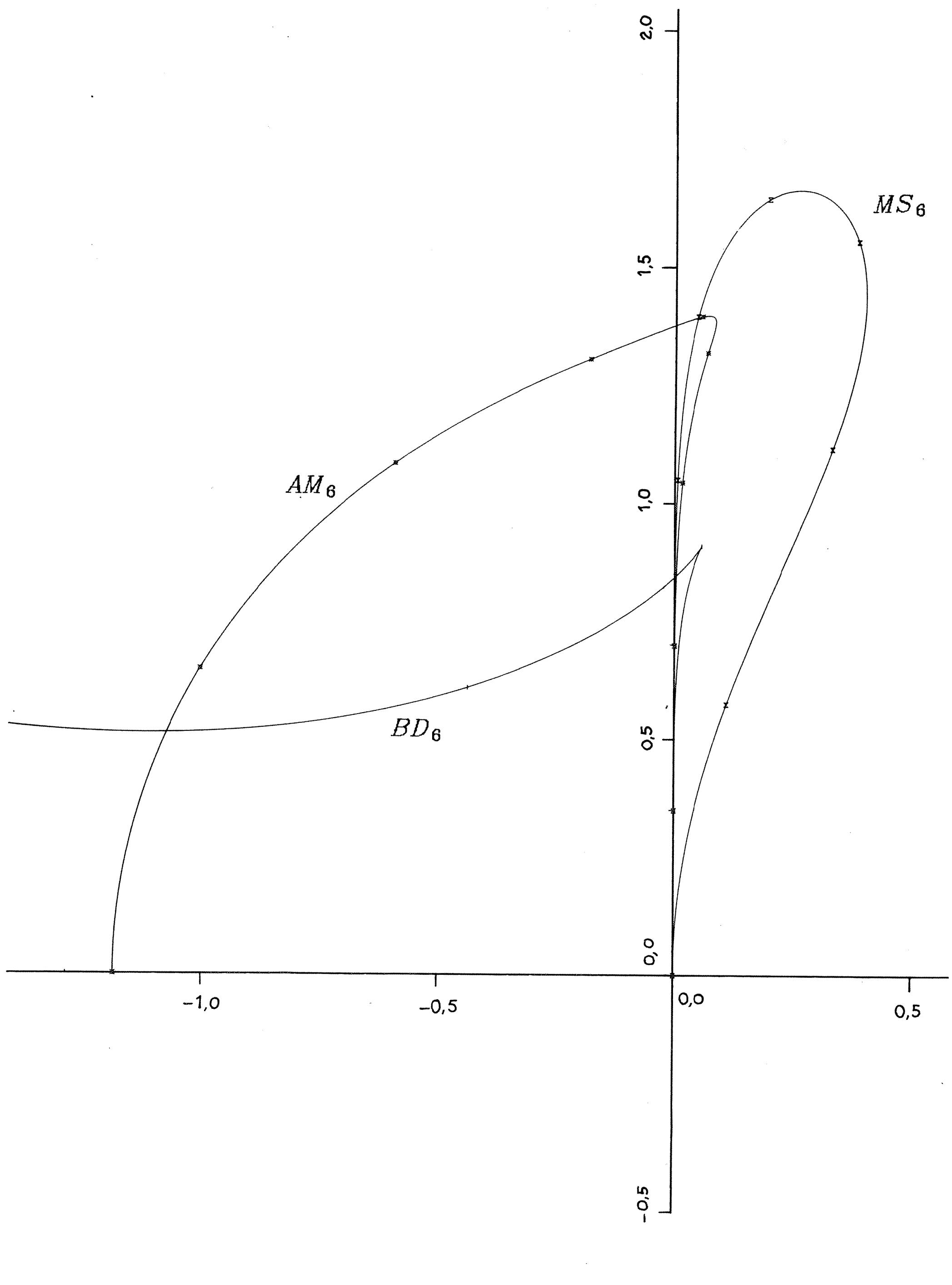


**Figure 3.1.** Stability regions of the $AM_6$, $MS_6$ and $BD_6$ method.

The stability regions of the corresponding minimax methods for realistic $(\underline{\nu},\bar{\nu})$-values (say $\underline{\nu}\leqslant\bar{\nu}\leqslant.1$) are very similar to the ones given in Figure 3.1. It turned out that there are no points on the imaginary axis for which the 6-th order Milne-Simpson method is absolutely stable.

In connection with stability we mention a paper by Skelboe and Christensen [8] in which the stability regions of the BD methods are enlarged by appending two exponential terms to the polynomial basis of the classical formulas.

## 4. Numerical comparisons

In this section the minimax methods generated by (2.9) are compared with the corresponding conventional linear multistep methods and with the methods based on the approach of Gautschi, that is, if $\omega_0$ is an estimate of the frequency of the exact local solution (cf. (2.4)) and $\nu_0=\omega_0 h$ then these methods are also based on (2.10) in which the condition (2.7a) is replaced by $\nu^{(l)}=l\nu_0$, $l=1,...,q$. These methods will be indicated by $AM_6(\nu_0)$, $MS_6(\nu_0)$ and $BD_6(\nu_0)$, respectively. In both cases the linear system (2.7b) (to obtain the coefficients for the linear multistep methods) was solved numerically. However, if the $\nu^{(l)}$-values are nearly equal this system in very ill-conditioned and we ran into numerical problems. In that case we changed to the system

$$\frac{d^j}{dz^j}\phi(z)|_{z=\frac{1}{2}i(\underline{\nu}+\bar{\nu})} = 0, \quad j=0,1,...,q-1.$$

In all experiments the starting values were taken from the exact solution or from a sufficiently accurate reference solution. The implicit relations were solved using Newton iteration. All problems were converted to their first-order equivalents and for measuring the obtained accuracy we used the number of correct significant digits in the end point $t_{end}$ of the integration, i.e.

$$sd := \log_{10}(L_2\text{−norm of the error at } \ t_{end}). \tag{4.1}$$

The calculations were performed an a CDC CYBER 175-750 which has a 48-bit mantissa yielding a machine precision of about 14 decimal digits.

Finally, we deliberately tried to select problems which are illustrative for the various kinds of difficulties we wanted to test for. The particular difficulty is mentioned in each subsection.

### 4.1. Periodic solutions

Consider the 6-th order *model* differential equation

$$\left\{\prod_{j=1}^{3}\left(\frac{d^2}{dt^2}+\omega_j\right)\right\}y(t) = 0, \quad 0\leqslant t\leqslant 12\pi=t_{end};\ \omega_j\geqslant 0 \tag{4.2}$$

with the exact solution

$$y(t) = \sum_{j=1}^{3}(C_j^+e^{i\omega_j t}+C_j^-e^{-i\omega_j t}), \tag{4.3}$$

where the constants $C_j^{\pm}$ are determined by the initial conditions.

Choosing

$$C_j^+=(1-i)/2,\ C_j^-=(1+i)/2,\ \omega_1=.7,\ \omega_2=\frac{2.8}{3},\ \omega_3=1.4 \tag{4.4}$$

we have the solution

$$y(t) = \sum_{j=1}^{3}(\sin(\omega_j t)+\cos(\omega_j t)) \tag{4.5}$$

which is periodic with frequency

$$\omega_0 = \frac{.7}{3} = .2333\ldots . \tag{4.6}$$

Applying the several methods we obtained the results as listed in Table 4.1. For this linear model problem, the theory of section 2 is confirmed rather well.

**Table 4.1.** *sd*-values obtained for problem {(4.2), (4.3), (4.4)}

| $h$ | $AM_6$ | $MS_6$ | $BD_6$ | $AM_6(\frac{.7}{3}h)$ | $MS_6(\frac{.7}{3}h)$ | $BD_6(\frac{.7h}{3}h)$ | $AM_6(.7h,1.4h)$ | $MS_6(.7h,1.4h)$ | $BD_6(.7h,1.4h)$ |
|---|---|---|---|---|---|---|---|---|---|
| $\pi/10$ | 1.44 | 1.97 | 0.41 | 1.62 | 2.13 | 0.59 | 3.12 | 3.56 | 2.09 |
| $\pi/25$ | 3.86 | 4.32 | 2.85 | 4.05 | 4.51 | 3.04 | 5.54 | 6.00 | 4.35 |
| $\pi/50$ | 5.66 | 6.12 | 4.66 | 5.85 | 6.31 | 4.85 | 7.34 | 7.80 | 6.34 |

We repeated the experiment but now the frequency $\omega_2$ was changed to 0.9. The solution in no longer periodic in the interval of integration, but we can regard it as "almost" periodic with frequency $\omega_0 \approx 0.23$. The results obtained differ only slightly from the results of Table 4.1 (there was no difference in *sd*-values found, greater than 0.03).

*Conclutions:*

- The change from a periodic solution to an "almost" periodic solution has no significant influence on the accuracy of the results
- The methods have some benefit from the Gautschi-approach; however, a substantial gain in accuracy is obtained by minimizing the local truncation error on the $\omega$-interval [.7, 1.4]
- Making a mutual comparison between the methods, the Milne-Simpson method seems to be the most accurate one for this problem (cf. Table 2.2).

### 4.2. Uncertainty in the periodicity

Next, we test the problem (cf. [2] and [7])

$$\ddot{y}(t) + (100 + \frac{1}{4t^2})y(t) = 0, \quad 1 \leq t \leq 10, \tag{4.7}$$

with the initial values according to the "almost" periodic particular solution

$$y(t) = \sqrt{t}J_0(10t), \tag{4.8}$$

where $J_0$ is a Bessel function of the first kind. Clearly, the frequency of this "almost" periodic solution is close to 10 and therefore we applied the Gautschi-methods with $\omega_0 = 10$. However, this problem is an example for which the spectrum of the Jacobian matrix gives detailed information about the local behaviour of the solution. A straightforward calculation reveals that the eigenvalues $\omega$ are approximately given by $\omega_{\pm}(t) \approx \pm 10i(1 + \frac{1}{800t^2})$. Hence, we applied our minimax methods with $\underline{\omega} = 9.9$ and $\bar{\omega} = 10.1$.

Table 4.2 shows the results of the various methods. Compared with the conventional methods there is a gain in accuracy of about two decimal digits in favour of the Gautschi-approach.

**Table 4.2.** $sd$-values for problem $\{(4.7), (4.8)\}$

| $h$ | $AM_6$ | $MS_6$ | $BD_6$ | $AM_6(10h)$ | $MS_6(10h)$ | $BD_6(10h)$ | $AM_6(9.9h,10.1h)$ | $MS_6(9.9h,10.1h)$ | $BD_6(9.9h,10.1h)$ |
|---|---|---|---|---|---|---|---|---|---|
| 1/25 | 2.27 | 2.02 | 1.05 | 4.50 | 4.51 | 3.32 | 7.20 | 5.66 | 6.42 |
| 1/50 | 4.57 | 5.14 | 3.24 | 6.89 | 6.80 | 5.56 | 8.60 | 8.73 | 7.74 |
| 1/100 | 6.38 | 6.73 | 5.49 | 8.46 | 8.88 | 7.66 | 10.30 | 10.77 | 9.30 |

The minimax methods, however, have a further increase in accuracy of about two decimal digits.

Finally, we anticipate that the accuracy can be still more increased by exploiting the special structure of the second order differential equation, i.e. the absence of the first order derivative. For, the ideas of minimizing the function $\phi$ on a suitable interval can analogously be applied to linear multistep methods which are designed for this type of equations, e.g. Strömer type methods.

### 4.3. Non-imaginary noise

In this subsection we want to test, apart from over- or underestimating the frequency of the solution, the influence of non-imaginary noise. By this, we mean that the local solution contains not only oscillatory components but also some "noise", caused by non-imaginary eigenvalues of the Jacobian matrix, i.e.

$$y(t) = \sum_j c_j e^{i\omega_j t} + \delta z(t). \tag{4.9}$$

For that purpose we selected the orbit equation (cf. Hull et. al. [3], Problem Class D)

$$\begin{aligned} &\ddot{u}(t)+u(t)/r^3=0, \;\; u(0)=1-\epsilon, \;\; \dot{u}(0)=0, \\ &\ddot{v}(t)+v(t)/r^3=0, \;\; v(0)=0, \;\; \dot{v}(0)=((1+\epsilon)/(1-\epsilon))^{1/2}, \\ &r^2=u^2(t)+v^2(t), \;\; 0\leq t\leq 12\pi \end{aligned} \tag{4.10}$$

with solution

$$\begin{aligned} &u(t)=\cos\tau-\epsilon, \;\; \dot{u}(t)=-\sin\tau/(1-\epsilon\cos\tau), \\ &v(t)=(1-\epsilon^2)^{1/2}\sin\tau, \;\; \dot{v}(t)=(1-\epsilon^2)^{1/2}\cos\tau/(1-\epsilon\cos\tau), \end{aligned} \tag{4.11}$$

where $\tau-\epsilon\sin\tau=t$ ($\epsilon$ is the eccentricity of the orbit).

The initial conditions correspond to a local solution of the form (4.9) in which $\delta$ is small.

First, we concentrate on an adequate treatment of the oscillatory part of the solution. For $\epsilon=0$, the complex eigenvalues of the Jacobian matrix are $\pm i$. However, for a non-zero eccentricity $\epsilon$ they are time-dependent and hard to determine in advance. For $\epsilon=.01$ we integrated this problem twice:

(i) the estimate of the frequency of the solution is 1, that is the Gautschi-approach was applied with $\omega_0=1$ and the minimax methods employed the $\omega$-interval [.9, 1.1]. The results are given in Table 4.3

(ii) secondly, $\omega_0=.9$ in the Gautschi-approach and the $\omega$-interval [.8, 1] for the minimax methods was used. Table 4.4 shows the results of this experiment.

From these tables we see a dramatic drop in accuracy for the Gautschi-methods when the frequency is wrongly estimated by only a small percentage.

**Table 4.3.** Results for problem $\{(4.10), (4.11), \epsilon=.01\}$ with $\omega_0=1.0$, $\underline{\omega}=.9$ and $\overline{\omega}=1.1$

| $h$ | $AM_6$ | $MS_6$ | $BD_6$ | $AM_6(h)$ | $MS_6(h)$ | $BD_6(h)$ | $AM_6(.9h,1.1h)$ | $MS_6(.9h,1.1h)$ | $BD_6(.9h,1.1h)$ |
|---|---|---|---|---|---|---|---|---|---|
| $\pi/10$ | 1.46 | 0.56 | 0.27 | 6.32 | 3.56 | 4.59 | 2.76 | 1.21 | 1.86 |
| $\pi/25$ | 4.34 | 3.09 | 3.08 | 7.68 | 5.69 | 6.73 | 5.01 | 3.69 | 4.04 |
| $\pi/50$ | 6.81 | 5.08 | 5.33 | 9.42 | 7.66 | 8.85 | 6.79 | 5.68 | 5.80 |

**Table 4.4.** Results for problem $\{(4.10), (4.11), \epsilon=.01\}$ with $\omega_0=.9$, $\underline{\omega}=.8$ and $\underline{\omega}=1.0$

| $h$ | $AM_6$ | $MS_6$ | $BD_6$ | $AM_6(.9h)$ | $MS_6(.9h)$ | $BD_6(.9h)$ | $AM_6(.8h,h)$ | $MS_6(.8h,h)$ | $BD_6(.8h,h)$ |
|---|---|---|---|---|---|---|---|---|---|
| $\pi/10$ | 1.46 | 0.56 | 0.27 | 0.94 | 0.74 | -.24 | 2.70 | 1.13 | 1.80 |
| $\pi/25$ | 4.34 | 3.09 | 3.08 | 3.73 | 3.06 | 2.55 | 4.94 | 3.62 | 3.97 |
| $\pi/50$ | 6.81 | 5.08 | 5.33 | 5.84 | 5.01 | 4.65 | 6.71 | 5.61 | 5.73 |

Again, the minimax methods show that they perform equally well in both cases and do not need an accurate foreknowledge of the frequency of the solution.

Let us return to the real subject of this subsection. The first part of the right-hand side of (4.9) is properly treated by the minimax methods (cf. (1.3)), the second part is not. If $h$ decreases, the influence of this term on the accuracy increases, hence the minimax methods gradually lose their superiority, as can be seen in Table 4.4. This effect is pronounced in the experiment with an eccentricity $\epsilon=.1$, the results of which can be found in Table 4.5.

**Table 4.5.** Results for problem $\{(4.10), (4.11), \epsilon=.1\}$ with $\omega_0=.9$, $\underline{\omega}=.8$ and $\overline{\omega}=1.0$

| $h$ | $AM_6$ | $MS_6$ | $BD_6$ | $AM_6(.9h)$ | $MS_6(.9h)$ | $BD_6(.9h)$ | $AM_6(.8h,h)$ | $MS_6(.8h,h)$ | $BD_6(.8h,h)$ |
|---|---|---|---|---|---|---|---|---|---|
| $\pi/10$ | 1.10 | -.64 | 0.09 | 0.90 | 0.31 | -.25 | 1.71 | -.47 | 0.78 |
| $\pi/25$ | 3.63 | 1.61 | 3.28 | 3.81 | 2.11 | 2.58 | 3.62 | 1.73 | 2.83 |
| $\pi/50$ | 5.14 | 3.61 | 4.25 | 6.34 | 4.09 | 4.87 | 5.25 | 3.73 | 4.31 |

## 4.4. Stiff components

So far, the $BD_6$ methods turned out to be inferior to the $AM_6$ and $MS_6$ methods as far as accuracy was concerned. In this subsection we will illustrate the use of $BD_6$ methods when the exact local solution is of the form (cf. (1.3))

$$y(t) \approx c_0 + \sum_{j=1}^{m_1} c_j e^{i\omega_j t} + \sum_{j=1}^{m_2} d_j e^{-\hat{\omega}_j t}, \tag{4.12}$$

where the $\hat{\omega}_j$ are positive and large (the so-called *stiff components).*

Apart from the initial phase, these components hardly influence the oscillatory behaviour of the solution but they demand for a highly stable method. For example, we see from figure 3.1 that the step size in the $AM_6$ method should satisfy $h \leq 1.18 / \hat{\omega}$, $\hat{\omega} = \max_j \hat{\omega}_j$, and that the $MS_6$ method is absolutely ustable for every $h$. However, in case of the $BD_6$ method, the value of $\hat{\omega}$ does not impose a restriction on the step size (see also [4]).

Let us consider the problem

$$\dddot{y}(t) + (2\epsilon y(t) - \lambda)\ddot{y}(t) + (1 + \epsilon^2 y^2(t) - 2\epsilon\lambda y(t))\dot{y}(t) - \lambda(1 + \epsilon^2 y^2(t))y(t) = \cos t, \qquad 0 \leq t \leq 20 \tag{4.13}$$

with initial conditions

$$y(0) = 1, \quad \dot{y}(0) = 1, \quad \ddot{y}(0) = -1. \tag{4.14}$$

For small values of $\epsilon$, the eigenvalues of the Jacobian matrix are approximately given by $\pm i$ and by

**Table 4.6.** Results of the $BD_6$ methods for problem {(4.13), (4.14)} with $\omega_0 = 1$, $\underline{\omega} = .9$ and $\bar{\omega} = 1.1$

| $h$ | $BD_6$ | $BD_6(h)$ | $BD_6(.9h, 1.1h)$ |
|---|---|---|---|
| 1/10 | 5.40 | 6.08 | 7.34 |
| 1/25 | 7.76 | 8.44 | 9.51 |

$\lambda$. In our experiment we choose $\epsilon = 10^{-2}$, $\lambda = -100$ and determined a reference solution with an explicit Runge-Kutta method using a very small step size. The results of the $BD_6$ methods are given in Table 4.6. For the step sizes of this table the $AM_6$ and $MS_6$ methods behaved unstable. Again, the minimax method is superior to the Gautschi-approach and to the conventional method.

## References


[1] Bettis, D.G., *Numerical integration of products of Fourier and ordinary polynomials,* Numer. Math. , Vol.14, 1970, pp. 421-434.

[2] Gautschi, W., *Numerical integration of ordinary differential equations based on trigonometric polynomials,* Numer. Math., Vol 3, 1961, pp. 381-397.

[3] Hull, T.E., W.H. Enright, B.M. Fellen and A.E. Sedgwick, *Comparing numerical methods for ordinary differential equations,* SIAM J. Numer. Anal., Vol. 9, 1972, pp. 603-637.

[4] Lambert, J.D., *Computational methods in ordinary differential equations,* J. Wiley, London, 1973.

[5] Liniger, W. and R.A. Willoughby, *Efficient numerical integration of stiff systems of ordinary differential equations,* SIAM J. Numer. Anal., Vol. 7, 1970, pp. 47-66.

[6] Lyche, T., *Chebyshevian multistep methods for ordinary differential equations,* Numer. Math, Vol. 19, 1972, pp. 65-72.

[7] Neta, B. and C.H. Ford, *Families of methods for ordinary differential equations based on trigonometric polynomials,* to appear in JCAM.

[8] Skelboe, S. and B. Christensen, *Backward differentiation formulas with extended regions of absolute stabilily,* BIT, Vol 21, 1981, pp. 221-231.

[9] Stiefel, E. and D.G. Bettis, *Stabilization of Cowell's method,* Numer. Math., Vol. 13, 1969, pp. 154-175.